Application of the Galerkin Method to the Prandtl Lifting Line Equation

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The method of Galerkin for obtaining approximate solutions of mathematical systems is applied to the singular linear integro-differential equation that is the analytical basis for the Prandtl wing theory. The approximating functions are the spanwise distributions, which were originally introduced by Betz. Although the general method is only outlined, the first- and second-order approximations are derived in complete detail. The first-order approximation is equivalent to an integrated or averaged elliptic wing, and it is shown that the second-order approximation yields numerical results for the case of constant spanwise properties and low-aspect ratio which agree quite well with the approximate values calculated by the method of Multhopp. The principal advantage of the Galerkin method is its inherent ability to average spanwise the local variations in wing properties.

I. Introduction

THE mathematical essence of the lifting surface theory due to Prandtl[1] is contained in the linear integro-differential system

$$P(\eta) = \Gamma(\eta) - \frac{1}{2}a(\eta)c(\eta) \times \left\{U\alpha(\eta) + \frac{1}{4\pi s}\int_{-1}^{1}\frac{d\Gamma(\xi)}{d\xi}\frac{d\xi}{\xi-\eta}\right\} = 0 \qquad (1.1)$$

where the associated boundary conditions are

$$\Gamma(\pm 1) = 0 \qquad (1.2)$$

Unfortunately, a serious difficulty in obtaining a solution is encountered because of the singularity at $\xi = \eta$, but regardless of the difficulty, many quite successful methods for the numerical solution of the equation have been developed.‡ The more frequently used methods fall into two classes: the point-fitting methods that culminate in the work of Multhopp,[4] and the integral methods, such as those of Gates[5] and Ziller,[6] who essentially use the Rayleigh-Ritz method at different steps in their treatments.

Although the point methods and, in particular, the elegant one due to Multhopp often have certain computational advantages, the integral methods enjoy a conceptual preference by employing the characteristics along the entire span or, at least, throughout strips, rather than using the properties only at fixed points. Moreover, the integral methods have a significant advantage whenever one wishes to compute the effects of flaps, control surfaces, or similar devices. Of course, the relative advantages of the two classes of methods will depend, in the last analysis, on the aerodynamic data one wishes to compute in a specific instance and on whatever computational equipment one has access to at what cost.

In view of the forementioned occasional advantages of the integral methods, it seems worthwhile to examine the possible application of the simplest integral method, namely, the one due to Galerkin,[7] to the Prandtl equation.

II. General Prescription

Since the Galerkin method is an unusual one, a brief description, with physical concepts instead of elaborate mathematical notations, may be justified. The central idea is the observation that the solution $f(x)$ to a boundary value problem $F[f(x)] = 0$, with constraints $G[f(x)] = 0$, can be approximated frequently by a series $\Sigma c_i f_i \sim f(x)$, where $G[\Sigma c_i f_i] = G[f(x)] = 0$ and the f_i are, consequently, virtual displacements and where the c_i are subsequently determined from the requirement of the vanishing of the virtual work $\int f_i(x)F[\Sigma c_i f_i(x)]dx = 0$, since $F[\Sigma c_i f_i]$ is the generalized force. The method is abstractly equivalent to the Rayleigh-Ritz method, although it is conceptually simpler and is often easier to use.

The first requirement in applying the Galerkin method to the Prandtl or another equation is to approximate the desired solution by a series whose terms are products of undetermined constants γ_i and functions $\Gamma_i(\eta)$ which satisfy the boundary conditions (1.2), and, although it is not a mathematical requirement, numerical feasibility almost requires that the mathematical expressions arising from the substitution of the approximating series into the equation should readily be put into closed form after a finite amount of work.

In the wing problem the requirement and the practical consideration are satisfied by the choice

$$\Gamma(\eta) = \sum_{i=0}^{k}\gamma_i\Gamma_i(\eta) = \sum_{i=0}^{k}\gamma_i\eta^{2i}(1-\eta^2)^{1/2} \qquad (2.1)$$

where the $\Gamma_i(\eta)$ are the functions used by Betz[8] when he first attacked the present problem by equating coefficients after expanding $\Gamma(\eta)$ into the same series. It is to be noted that an entirely different purpose is being served in the present situation.

It is obvious that the requirement $\Gamma(\pm 1) = 0$ is satisfied because of the factor $(1 - \eta^2)^{1/2}$ in each term; on the other hand it is not quite so obvious that the integrals

$$\int_{-1}^{1}\frac{d[\xi^{2i}(1-\xi^2)^{1/2}]}{d\xi}\frac{d\xi}{\xi-\eta} \qquad (2.2)$$

are expressible in closed form. However, if the substitutions $\xi = \cos\vartheta$ and $\eta = \cos\varphi$ are made, and if one notes that $\cos^{2j+1}\vartheta$ may be expressed by a finite sum of the form

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‡ Excellent summaries are given by Thwaites[2] and by Robinson and Laurmann,[3] although their viewpoints are different.

$\sum_{j=0}^{p} A_j \cos(2j+1)\vartheta$, then the integrals may be reduced to a finite sum of terms with the structure

$$\int_0^\pi \frac{\cos(2j+1)\vartheta}{\cos\vartheta - \cos\varphi}\,d\vartheta = \frac{\pi \sin(2j+1)\varphi}{\sin\varphi} \tag{2.3}$$

wherein the Cauchy principal value has been taken. A simple derivation of Eq. (2.3) has been given by Glauert.[9]

Now that the preliminaries are done, one puts the series from Eq. (2.1) into $P(\eta)$ in Eq. (1.1) and obtains

$$P[\Gamma(\eta)] = P(\gamma_0, \gamma_1, \ldots, \gamma_k, \eta) \tag{2.4}$$

Following the idea of Galerkin, one then seeks an approximate or average solution so that

$$\int_{-1}^{1} \Gamma_j(\eta) P(\gamma_0\ \gamma_1, \ldots, \gamma_k, \eta)d\eta = \sum_{i=0}^{k} a_{ij}\gamma_i + b_j = 0 \tag{2.5}$$

where a_{ij} and b_j are constants and $i, j = 0, 1, \ldots, k$. The linear algebraic equations (2.5) are subsequently solved for γ_i by Cramer's rule or another scheme, provided that *det* $a_{ij} \neq 0$ and some $b_j \neq 0$ which should be verified in each specific calculation. Under certain hypotheses the Galerkin process may be shown to converge as $k \to \infty$ to a completely valid solution of the original system.[10]

III. First-Order Approximation or the Equivalent Elliptic Wing

As a first-order approximation one takes only a single term in the series or the following expression:

$$\Gamma(\eta) = \gamma_0\Gamma_0(\eta) = \gamma_0(1-\eta^2)^{1/2} \tag{3.1}$$

which amounts to the same thing as finding an equivalent elliptic wing.§ The substitution of Eq. (3.1) into Eq. (1.1) yields

$$P[\gamma_0\Gamma_0(\eta)] = \gamma_0(1-\eta^2)^{1/2} - \tfrac{1}{2}a(\eta)c(\eta)[U\alpha(\eta) - (\gamma_0/4s)] \tag{3.2}$$

since it follows from Eq. (2.3) that

$$\int_{-1}^{1} \frac{\xi d\xi}{(1-\xi^2)^{1/2}(\xi-\eta)} = -\pi \tag{3.3}$$

After multiplying Eq. (3.2) by $(1-\eta^2)^{1/2}$, integrating with respect to η from -1 to 1, and equating the integrated result to zero, a slight manipulation gives

$$\gamma_0 = \frac{12sU\int_{-1}^{1}(1-\eta^2)^{1/2}a(\eta)c(\eta)\alpha(\eta)d\eta}{32s + 3\int_{-1}^{1}(1-\eta^2)^{1/2}a(\eta)c(\eta)d\eta} \tag{3.4}$$

Whenever $a(\eta)$, $c(\eta)$, and $\alpha(\eta)$ are constants along the span and are equal to a, c, and α, respectively, Eq. (3.4) reduces to

$$\gamma_0 = \frac{12\pi s\alpha acU}{64s + 3\pi ac} \tag{3.5}$$

since $\int_{-1}^{1}(1-\eta^2)^{1/2}d\eta = \pi/2$.

It is important to note that spanwise variations in $a(\eta)$, $c(\eta)$, and $\alpha(\eta)$ are taken into account easily by two simple numerical integrations. In particular, if $a(\eta)$, $c(\eta)$, and $\alpha(\eta)$ can be expressed as powers of η, the integrals in Eq. (3.4) can be given in closed form, since $\int_{-1}^{1}\eta^p(1-\eta^2)^{1/2}d\eta$ can be integrated explicitly and is given in the common integral tables[16]; moreover, if strip theory is used, i.e., $a(\eta) = a_i$, $c(\eta) = c_i$, and $\alpha(\eta) = \alpha_i$ for $\eta_i \leqslant \eta \leqslant \eta_{i+1}$, the integrals in Eq. (3.4) can be given once again in closed form since

$$2\int_a^b (1-\eta^2)^{1/2}d\eta = b(1-b^2)^{1/2} - a(1-a^2)^{1/2} + \sin^{-1}b - \sin^{-1}a$$

§ The notion of an equivalent elliptic wing has been previously used by Helmbold,[11] Scholz,[12] Diederich,[13, 14] and Schrenk.[15]

IV. Second Approximation

If one takes two terms in Eq. (2.1), the approximating function becomes

$$\Gamma(\eta) = \gamma_0\Gamma_0(\eta) + \gamma_1\Gamma_1(\eta) = \gamma_0(1-\eta^2)^{1/2} + \gamma_1\eta^2(1-\eta^2)^{1/2} \tag{4.1}$$

and the subsequent substitution into Eq. (1.1) gives

$$P(\eta) = \gamma_0(1-\eta^2)^{1/2} + \gamma_1\eta^2(1-\eta^2)^{1/2} - \frac{1}{2}Ua(\eta)c(\eta)\alpha(\eta) + \frac{\gamma_0 a(\eta)c(\eta)}{8s} - \frac{\gamma_1 a(\eta)c(\eta)}{8\pi s}\times \int_{-1}^{1}\frac{(2\xi - 3\xi^3)d\xi}{(1-\xi^2)^{1/2}(\xi-\eta)} \tag{4.2}$$

The definite integral in Eq. (4.2) can be evaluated by the process that was outlined in Sec. II. After inserting the value of the integral $(1-6\eta^2)\pi/2$, the averaging process of the Galerkin method may be carried out by multiplying $P(\eta)$ in Eq. (4.2) by $\Gamma_0(\eta)$ and then integrating with respect to η from -1 to 1 to obtain one equation and by multiplying $P(\eta)$ by $\Gamma_1(\eta)$ and again integrating from -1 to 1 to obtain a second equation. These two equations are

$$\gamma_0\left[\frac{4}{3} + \frac{1}{8s}\int_{-1}^{1}(1-\eta^2)^{1/2}a(\eta)c(\eta)d\eta\right] + \gamma_1\left[\frac{4}{15} + \frac{1}{16s}\int_{-1}^{1}(6\eta^2-1)(1-\eta^2)^{1/2}a(\eta)c(\eta)d\eta\right] = \frac{1}{2}U\int_{-1}^{1}(1-\eta^2)^{1/2}a(\eta)c(\eta)\alpha(\eta)d\eta \tag{4.3}$$

and

$$\gamma_0\left[\frac{4}{15} + \frac{1}{8s}\int_{-1}^{1}\eta^2(1-\eta^2)^{1/2}a(\eta)c(\eta)d\eta\right] + \gamma_1\left[\frac{4}{35} + \frac{1}{16s}\int_{-1}^{1}\eta^2(6\eta^2-1)(1-\eta^2)^{1/2}a(\eta)c(\eta)d\eta\right] = \frac{1}{2}U\int_{-1}^{1}\eta^2(1-\eta^2)^{1/2}a(\eta)c(\eta)\alpha(\eta)d\eta \tag{4.4}$$

In the case of constant spanwise characteristics, the equations simplify considerably, and they become

$$\gamma_0[(4/3) + ac\pi/(16s)] + \gamma_1[(4/15) + ac\pi/(64s)] = Uac\alpha\pi/4 \tag{4.5}$$

and

$$\gamma_0[(4/15) + ac\pi/(64s)] + \gamma_1[(4/35) + ac\pi/(64s)] = Uac\alpha\pi/16 \tag{4.6}$$

The direct application of Cramer's rule gives

$$\gamma_0 = \frac{300s\alpha Uk(4+63k)}{128 + 1980k + 4725k^2} \tag{4.7}$$

and

$$\gamma_1 = \frac{1680s\alpha Uk}{128 + 1980k + 4725k^2} \tag{4.8}$$

where

$$k = ac\pi/(64s) \tag{4.9}$$

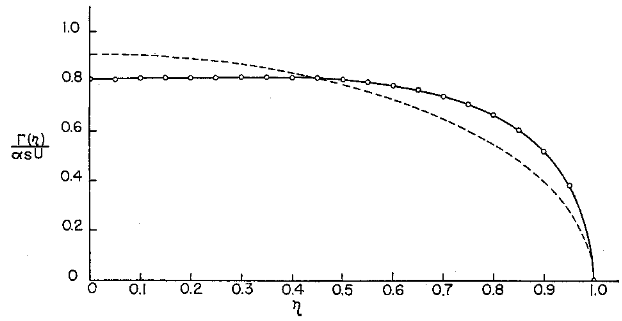

Fig. 1 Numerical values for the spanwise distribution of circulation. Dashed line shows the value of $\Gamma(\eta)/(\alpha s U)$ for the particular case $ac = 2s$, as calculated from the first-order Galerkin approximation; solid line shows the values for the second-order Galerkin approximation. Circles indicate equidistant numerical values for the same aerodynamic situation as calculated by the classical trigonometrical method of fitting at fixed points due to Multhopp.

The spanwise variations of $a(\eta)$, $c(\eta)$, and $\alpha(\eta)$ may be treated by an obvious generalization of the procedure for the first approximation; the only difference is the more complicated integrations occurring in the coefficients of γ_i in Eqs. (4.3) and (4.4), although all of the definite integrals can still be expressed in closed form.

V. A Simple Example

An example that is often used for comparative purposes is the one of a wing at constant incidence for which $ac/(8s) = \mu = \frac{1}{4}$ or $ac = 2s$, and the numerically calculated quantity that is to be compared is $\Gamma(\eta)/(\alpha s U)$. Figure 1 shows the first- and second-order approximations for $\Gamma(\eta)/(\alpha s U)$ as computed by the Galerkin method and compares them with the approximation that was obtained by the Multhopp method.[17] The maximum difference in the second-order Galerkin approximation and the Multhopp values is 0.5%; the difference in computational work is large, since the Galerkin method, including the evaluation of the Betz functions, takes about 15 min with a desk calculator.

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